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On Birkhoff Quadrature Formulas II

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The following problem was raised by P. Turan [5]. Let

$$-1 = x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} = 1 \tag{1.1}$$

be the zeros of

$$\pi_n(x) = (1-x^2)\, P'_{n-1}(x), \tag{1.2}$$

where $P_n(x)$ is the Legendre polynomial of degree n. Let f be an arbitrary polynomial of degree $\leqslant 2n-1$. For what choice of λ_{kn}, μ_{kn} $(k=1, 2, ..., n)$ do we have

$$\int_{-1}^{1} f(x)\, dx = \sum_{k=1}^{n} \lambda_{kn} f(x_{kn}) + \sum_{k=1}^{n} \mu_{kn} f''(x_{kn})? \tag{1.3}$$

The above problem was solved by one of us [6]. Later in 1986, the first author [7] even gave a very simple method of determining λ_{kn}, μ_{kn} which makes no use of Turan fundamental polynomials of (0, 2) interpolation based on the nodes (1.2). It is very natural to raise the following generalization of the Turan problem. Let f be an arbitrary polynomial of degree $\leqslant 2n-1$; determine λ_{jnm}, μ_{knm}, $k=1, 2, ..., n$, such that

$$\int_{-1}^{1} f(x)\, dx = \sum_{k=1}^{n} \lambda_{knm} f(x_{kn}) + \sum_{k=1}^{n} \mu_{knm} f^{(m)}(x_{kn}). \tag{1.4}$$

When $m=2$, we observed that $\lambda_{kn} \geqslant 0$, $\mu_{kn} \geqslant 0$, $k=1, 2, ..., n$.

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It is worthy to investigate whether a similar situation prevails in which m is an even positive integer.

The object of this paper is to consider the above problem when $m=3$ and $m=4$ with the following modifications. Here we will prescribe Birkhoff data at the internal nodes and Hermite data at the end points ± 1. More precisely we shall prove:

THEOREM 1. *Let f be any given polynomial of degree $\leqslant 2n-1$. Then for $n \geqslant 4$ there exists a unique choice of A_n, B_n, C_n, D_n such that*

$$\int_{-1}^{1} f(x)\,dx = (f(1)+f(-1))\,A_n + B_n \sum_{k=2}^{n-1} \frac{f(x_{kn})}{P_{n-1}^2(x_{kn})} + C_n(f'(1)-f'(-1)) + D_n \sum_{k=2}^{n-1} \frac{x_{kn}(1-x_{kn}^2)\,f'''(x_{kn})}{P_{n-1}^2(x_{kn})}, \tag{1.5}$$

where the x_{kn}'s are the zeros of $\pi_n(x)$ given by (1.3). *Further*

$$A_n = \frac{8n^2-25n+24}{n(2n-1)(2n^2-8n+9)}, \qquad B_n = \frac{4(n-2)(2n-3)}{n(2n-1)(2n^2-8n+9)} \tag{1.6}$$

$$C_n = \frac{-1}{(2n-1)(2n^2-8n+9)}, \qquad D_n = \frac{1}{n(n-1)(2n-1)(2n^2-8n+9)}. \tag{1.7}$$

We shall also prove:

THEOREM 2. *Let f be an arbitrary polynomial of degree $\leqslant 2n-1$. Then for $n \geqslant 6$ there exists a unique choice of a_n, b_n, c_n, d_n such that*

$$\int_{-1}^{1} f(x)\,dx = (f(1)+f(-1))\,a_n + b_n \sum_{k=2}^{n-1} \frac{f(x_{kn})}{P_{n-1}^2(x_{kn})} + c_n(f'(1)-f'(-1)) + d_n \sum_{k=2}^{n-1} \frac{(1-x_{kn}^2)^2 f^{(\text{iv})}(x_{kn})}{P_{n-1}^2(x_{kn})}. \tag{1.8}$$

Moreover,

$$a_n = \frac{8n^3-48n^2+88n-60}{n(2n-1)(2n^3-13n^2+29n-24)} \tag{1.9}$$

$$b_n = \frac{8n^3-48n^2+94n-60}{n(2n-1)(2n^3-13n^2+29n-24)} \tag{1.10}$$

$$c_n = \frac{2}{(2n-1)(2n^3-13n^2+29n-24)} \tag{1.11}$$

$$d_n = \frac{-c_n}{4n(n-1)}. \tag{1.12}$$

From Theorem 1 the following corollary may be deduced.

COROLLARY. *Let $f \in c[-1, 1]$. Then for $n \geqslant 4$*

$$\lim_{n \to \infty} \left\{ (f(1) + f(-1)) A_n + B_n \sum_{k=2}^{n-1} \frac{f(x_{kn})}{P_{n-1}^2(x_{kn})} \right\} = \int_{-1}^{1} f(x)\, dx.$$

Concerning various aspects of Birkhoff quadrature, we may refer to the book of Lorentz *et al.* [3]. Another related work is due to Nevai and Varma [4].

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Proof of Theorem 1. Let f be an arbitrary polynomial of degree $\leqslant 2n-1$. This we shall denote by $f \in \pi_{2n-1}$. The following identity will play an important role in determining our quadrature formula (1.5):

$$\int_{-1}^{1} x(1-x^2) f'''(x)\, dx$$
$$= 2(f'(1) - f'(-1)) - 6(f(1) + f(-1)) + 6 \int_{-1}^{1} f(x)\, dx. \quad (2.1)$$

From the known result [2], it also follows that if $f \in \pi_{2n-3}$ then for $n \geqslant 2$ we have

$$\int_{-1}^{1} f(x) = \frac{2}{n(n-1)} \left[f(1) + f(-1) + \sum_{k=2}^{n-1} \frac{f(x_{kn})}{P_{n-1}^2(x_{kn})} \right]. \quad (2.2)$$

Next, we note that if $f \in \pi_{2n-3}$ we have (here we use (2.1) and (2.2) as well)

$$\sum_{k=2}^{n-1} \frac{x_{kn}(1-x_{kn}^2) f'''(x_{kn})}{P_{n-1}^2(x_{kn})} = \frac{n(n-1)}{2} \int_{-1}^{1} x(1-x^2) f'''(x)\, dx$$
$$= n(n-1)[f'(1) - f'(-1) - 3(f(1) + f(-1)) + 3 \int_{-1}^{1} f(x)\, dx]. \quad (2.3)$$

We can also rewrite (2.2) as

$$\sum_{k=2}^{n-1} \frac{f(x_{kn})}{P_{n-1}^2(x_{kn})} = \frac{n(n-1)}{2} \int_{-1}^{1} f(x)\, dx - (f(1) + f(-1)). \quad (2.4)$$

It is now easy to complete the proof of (1.5). First let $f \in \pi_{2n-3}$. Then on using (2.3), (2.4) the rhs of (1.5) can be expressed by

$$(f(1)+f(-1))\,A_n+B_n\left\{\frac{n(n-1)}{2}\int_{-1}^{1}f(x)\,dx-(f(1)+f(-1))\right\}$$

$$+C_n(f'(1)-f'(-1))+D_n n(n-1)\left\{f'(1)-f'(-1)\right.$$

$$\left.-3(f(1)+f(-1))+3\int_{-1}^{1}f(x)\,dx\right\}$$

$$=(f(1)+f(-1))\{A_n-B_n-3n(n-1)\,D_n\}$$

$$+(C_n+n(n-1)\,D_n)(f'(1)-f'(-1))$$

$$+\int_{-1}^{1}f(x)\,dx\left\{\frac{n(n-1)}{2}B_n+3n(b-1)\,D_n\right\}.$$

On using (1.6), (1.7) we observe that

$$A_n-B_n-3n(n-1)\,D_n=0,\qquad C_n=-n(n-1)\,D_n,$$

and

$$B_n+6D_n=\frac{2}{n(n-1)}.$$

Therefore, the rhs of (1.5) is equal to $\int_{-1}^{1}f(x)\,dx$ if $f\in\pi_{2n-3}$.

Next, we will show that if $f_0(x)=\pi_n(x)\,P'_{n-1}(x)$ (1.5) still is valid. For this purpose we note the following observations:

$$f_0'(1)=\frac{-n^2(n-1)^2}{2},\qquad f_0'(-1)=\frac{n^2(n-1)^2}{2}$$

$$f_0'''(x_{kn})=\frac{12n^2(n-1)^2\,x_{kn}\,P_{n-1}^2(x_{kn})}{1-x_{kn}^2},\qquad k=2,3,...,n-1,$$

$$\int_{-1}^{1}f_0(x)\,dx=\frac{2n(n-1)}{2n-1},\qquad \sum_{k=2}^{n-1}\frac{1}{1-x_{kn}^2}=\frac{n(n-1)-2}{4}.$$

On substituting these expressions in (1.5) we obtain

$$\frac{2n(n-1)}{2n-1}=\int_{-1}^{1}f_0(x)\,dx=C_n(-n^2(n-1)^2)+D_n\sum_{k=2}^{n-1}\frac{12n^2(n-1)^2\,x_{kn}^2}{1-x_{kn}^2}.$$

In other words we must prove

$$\frac{2}{2n-1} = -C_n(n(n-1)) + 12n(n-1)\,D_n\left\{-(n-2) + \frac{n(n-1)-2}{4}\right\}$$

$$= -C_n(n(n-1)) - 12C_n\left\{\frac{n^2-5n+6}{4}\right\}$$

$$\frac{2}{2n-1} = -c_n\{4n^2-16n+18\}.$$

But this is valid in view of (1.7).

Finally, if f is any odd polynomial of degree $\leqslant 2n-1$ (1.5) is obviously valid. This proves Theorem 1. Proof of Theorem 2 is similar to Theorem 1, so we omit the details.

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